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MAXIMAL ELEMENTS AND SYMMETRY OF THE SEMIGROUP OF VALUES OF A CURVE SINGULARITY WITH SEVERAL BRANCHES

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In this paper C is a reduced, algebroid, plane curve and $C_1, C_2, \dots, C_d$ its branches. If v_i is the valuation associated with C_i denote by $\underline{v}: D(\mathcal{O}) \longrightarrow \mathbb{Z}_+^d$ the mapping given by $\underline{v}(h) = (v_1(h), \dots, v_d(h))$. ($D(\mathcal{O})$ is the set of nonzero divisors in the local ring $\mathcal{O}$ of C and $\mathbb{Z}_+ = \{ n \in \mathbb{Z} \ / \ n \geq 0 \}$). The semigroup of values of C, S(C) or S, is defined to be the additive subsemigroup $\operatorname{Im} \underline{v}$ of $\mathbb{Z}_+^d$. One has that C and C' are equisingular if and only if, for a suitable ordering on the set $\{1, \dots, d\}$ the semigroups S(C) and S(C') are the same (see [W]).

Let $I = \{1, 2, \dots, d\}$, $d \geq 2$, $J \subset I$, $J \neq \emptyset$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d = \mathbb{Z}_+^I$ and consider the following sets

$$\Delta_J(\alpha) = \{ \beta = (\beta_1, \dots, \beta_d) \in S \ / \ \beta_i = \alpha_i \ \forall i \in J, \ \beta_k > \alpha_k \ \forall k \notin J \}$$

$\Delta(\alpha) = \bigcup_{i=1}^{d} \Delta_{\{i\}}(\alpha)$. Denote by pr_J the natural projection $\mathbb{Z}_+^I \longrightarrow \mathbb{Z}_+^J$ and $S_J = pr_J S$. Finally " $\leq$ " will denote the ordering on $\mathbb{Z}_+^d$ given by

$$\alpha \leq \beta \iff pr_i \alpha \leq pr_i \beta \qquad \text{for all } i = 1, \dots, d$$

<u>Definition</u>.- $\alpha \in S$ will be said to be <u>maximal</u> of S if $\Delta(\alpha) = \emptyset$. If, moreover, $\Delta_J(\alpha) = \emptyset \ \ \forall J \subset I, \ J \neq I$, α will be said to be an <u>absolute maximal</u>. If α is maximal and $\Delta_J(\alpha) \neq \emptyset, \forall J \subset I$ with $\#J \geq 2$, α will be said to be <u>relative maximal.</u>

<u>Theorem (generation)</u>.- Let $R = \{ \alpha^1, \dots, \alpha^n \}$ be the set of relative maximal elements of S. Let $\beta \in \mathbb{Z}_+^d$ such that if $\#J = d-1$ then $pr_J(\beta) \in S_J$. Then $\beta \in S$ if and only if $\beta \notin \Delta(\alpha^i)$ for $i = 1, \dots, n$.

<u>Notas</u>: a) Above result is proved in [D-1] in an arithmetical way for the semigroups of (not necesarily plane) algebroid curves. b) R is a

finite set because of the existence of conductor in S. c) Since S_J is the semigroup of the curve $C_J = \bigcup_{i\in J} C_i$ the theorem provides a inductive process describing the semigroup S.

Definition.- The Apery basis, $A_\gamma(S)$, of S with respect to $\gamma \in S$ is defined to be the set of elements $\alpha \in S$ such that $\alpha-\gamma \notin S$. We will call principal elements of S with respect to $\gamma \in S$ the elements in the set

$$N_\gamma(S) = \{ \alpha \in S \ / \ \Delta(\alpha) \subset A_\gamma(S) \}.$$

Each maximal element is a principal one (relative to any γ) and if α is principal (relative to γ) then $pr_J(\alpha)$ is principal with respect to $pr_J(\gamma) + \xi^J$, where $\xi^J = pr_J(\underline{v}(\prod_{i\notin J} f_i))$, f_i being an equation for the i-th. branch.

Now, let $Q \in \mathbb{Z}_+^d$ be the element given by

$$pr_i Q = c_i + \sum_{\substack{j=1\\ j\neq i}}^{d} I_{ij} - 1,$$

where I_{ij} is the intersection multiplicity of C_i and C_j, c_i is the conductor of S_i. If $\alpha \in N_\gamma(S)$ one has that $\alpha \leq Q+\gamma$ and if α is maximal then $\alpha \leq Q$. The main result states as follows.

Theorem (Symmetry of maximal elements)

A) Q is a relative maximal of S. Moreover $Q + (1,\overset{d)}{\ldots},1)$ is the conductor of the semigroup S.

B) Let $\alpha \in S$. Then α is maximal if and only if $Q - \alpha \in S$. Moreover if $\alpha, \beta \in S$ and $\alpha + \beta = Q$ one has that α is absolute maximal if and only if β is relative maximal.

C) If $\gamma, \nu \in S$, then $\nu \in N_\gamma(S)$ if and only if $Q + \gamma - \nu \in S$.

The statement A) implies that $Q + \gamma$ is the maximum of the principal elements relative to γ.

Now if $\alpha, \beta \in \mathbb{Z}_+^d$ then one has:

If $\alpha + \beta = Q$, then α is maximal if and only if β is maximal.

If $\alpha + \beta = Q + \gamma$, then $\alpha \in N_\gamma(S)$ if and only if $\beta \in N_\gamma(S)$

For d = 1, these properties are well known (see [K]): If $\alpha + \beta = c - 1$ then $\alpha \in S \Leftrightarrow \beta \notin S$ and if $\alpha + \beta = c + \gamma - 1$ then $\alpha \in A_\gamma(S) \Leftrightarrow \beta \in A_\gamma(S)$, where c is the conductor of S.

In the proof of the Theorem ([D-2]) we will use induction on the number $d \geq 2$ of branches of C. If d = 2 this Theorem is essentially proved in [G].

The proof of the Theorem is based on the theorem of generation and the properties given before the statement. The proof proceeds by induction in such a way that both B) and C) appear within the same inductive process, where the joining of B) and C) is suggested by the fact "α is a maximal implies $pr_J(\alpha) \in N_{\xi J}(S_J)$".

From above theorems it follows that the absolute maximal elements of the semigroup S are enough for determining S if one knows the projected semigroups, $pr_J(S)$, where $\#J = d-1$. In fact , absolute and relative maximals are equivalent data bacause the symmetry Theorem.

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