EXTRACTA MATHEMATICAE 1, 15-17 (1986)

DESINGULARIZATION OF THREE -DIMENSIONAL VECTOR FIELDS

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Here we present some results obtained in [6, 7, 8, 9]. If F is a singular foliation over a variety X, dim X = 2, making a finite number of quadratic blowing-ups of X, one obtains a (saturated) foliation $F^{\sim}$ given at each singular point by a vector field $D = a\partial/\partial x + b\partial/\partial y$ whose linear part has an eigenvalue $\lambda \neq 0$ and the other root μ of the characteristic polynomial verifies $\mu/\lambda \notin \mathbb{Q}_+$. There are two problems: first to show that if $\nu = \min(\nu(a), \nu(b))$ then it decreases after a finite number of blowing-ups if $\nu \geq 2$, second to obtain the above special feature if $\nu = 1$; [6,7] deal with the first problem and [8,9] with the second one in the case dim X = 3.

1. Logarithmic viewpoint. An unidimensional distribution D over X is an invertible O_X-submodule of the tangent sheaf Θ_X. It defines an unidimensional singular foliation F_D over X. D is multiplicatively irreducible iff $D = \alpha(D)$, where $\alpha(D)$ is the double orthogonal. The saturation of F_D is sat $(F_D) = F_{\alpha(D)}$. P is a singular point of D iff $\nu_P(D) \geq 1$, $\nu_P(D)$ = order of D_P as a submodule of $\Theta_{X,P}$. The behaviour of $\nu_P(D)$ is not very good under quadratic blowing-ups. Moreover it is an important question to study the leaves of F_D through a singular point ([11], [2], [3] if dim X = 2). The strict transforms of these leaves are the leaves of $F^{\sim}$ not contained in the exceptioal divisor E. This allows us to introduce E. D is adapted to E iff $D \subset \Theta_X[E]$ = dual sheaf of the logarithmic forms $\Omega_X[\log E]$. D is m.i. and adapted to E iff $D = \alpha_E(D)$, where $\alpha_E(D)$ is obtained as above. The adapted order $\nu(D, E, P)$ is the order of D_P as a submodule of $\Theta_{X,P}[E]$. One has

Theorem 1. ([6]. I. (3.1.4); [4]. II. 1.3.3). If $\pi : X' \to X$ is a quadratic blowing-up and (X', E', D') is the strict transform of (X, E, D) ($E' = \pi^{-1}(E \cup \{center\})$), then $\nu(D', E', P') \leq \nu(D, E, \pi(P'))$ for each P'. The blowing-ups to be considered have center Y with normal crossings with E in order that E' would be a normalcrossings divisor. If Y verifies that $\nu(D', E', P') \leq \nu(D, E, \pi(P'))$ and dim $Y \leq$ dim $X-2$, then Y is weakly permissible.

2. Reduction games. Let us begin with (X, E, D, P) with $\nu(D,E,P) = r \geq 2$. We are player A. Choose a weakly permissible center Y at P and let $\pi : X' \longrightarrow X$ be the corresponding blowing-up. Assume that the player B chooses a point P' over P. If $\nu(D', E', P') \leq 1$, we have won. Otherwise the game begins at the status (X', E', D', P'). [6] is devoted to the proof of

Theorem 2. ([6] I. 4.2.9). If dim X = 3, then there is a strategy for the player A in order to win in a finite number of steps.

3. Global results. [7] is devoted to prove that for a special kind of singularities one can reduce the adapted order in a global manner. Let $J^r(f)$ = ideal of the strict tangent space of $cl^r(f)$ ($\nu(f) \geq r$, $cl^r(f)$ = = image in m_P^r / m_P^{r+1}). Let $W_o = G_r R$ and $W_i = J^r(D(H(W_{i-1} \cap Gr^1 R)))$ and set $W(D, E, P) = \cap W_i$. P is of the type zero iff $W(D, E, P) \neq 0$. The W-directrix $Dir_W(D,E,P)$ is the subvariety of $T_P X$ given by W(D,E,P). A weakly permissible curve Y is permissible iff $\nu_Y(D(I(Y))) \geq \rho$, $\nu_Y(D(I(P))) \geq \rho-1$, where $\rho = \min(r+1, \nu_P(D(I(Y))))$. The main results of [7] are the following ones.

Proposition 1. If Y is permissible, tangent to the W-directrix, $\pi : X' \to X$ is the blowing-up centered at Y and $\nu(D', E', P') = \nu(D, E, \pi(P'))$, then $P' \in Proj(Dir_W(D,E,P) / T_{\pi(P')}Y)$ and dim $Dir_W(D',E',P') \leqslant$ dim Dir (D, E, $\pi(P')$). ([7]. (2.4)).

Theorem 3. ([7] 3.1). Assume that r = biggest adapted order, $r \geq 2$, dim X = 3 and if $\nu(D,E,P) = r$ then P is of the type zero. Then there is a finite sequence of permissible blowing-ups such that the strict transform $(X^\sim, D^\sim, P^\sim)$ verifies $\nu(D^\sim, E^\sim, P^\sim) < r$ for each $P^\sim$.

4. Final forms. One has also the analogous of $\mu/\lambda \notin \mathbb{Q}_+$ for dim X = 3. Assume that $\nu(D, E, P) = 0$, P is regular iff $\nu(D, \emptyset, P) = 0$, P is a simple point iff it is not regular, $1 = e(E, P) = \#$ (components of E) and the characteristic polynomial of the linear part of a generator of D_P has roots $(1, \lambda, \mu)$ where $\lambda \notin \mathbb{Q}_+$, $\mu \notin \mathbb{Q}_+$. P is a simple corner iff it is not regular, $e(E,P) \geq 2$ and one can take a generator D of D_P and

$Y = E_1 \cap E_2$ (E_i = irr. comp. of E) veriying that the linear map $f \mapsto cl^1(D(f))$ in $In^1(I(Y))$ has roots $1, \lambda$, $\lambda \notin \mathbb{Q}_+$ for its characteristic polynomial. The main results of [9] are the following ones

Theorem 4. ([9]. 4.10). Let dim X = 3 and assume that $\nu(D,E,P) = 0$ for each P. Then after a finite number of blowing-ups, the strict transform ($X^\sim$, $E^\sim$, $D^\sim$) verifies that each closed point is a regular point, a simple point or a simple corner.

Corollary. ([9]. 7.5, also [5], [8]). Let π: $X^\sim \rightarrow X$ be the above morphism. Then there is a bijection ψ between the integral branches of D through a singular point P of D not contained in E and the simple points $P^\sim$ over P, such that $\psi(\Gamma)$ = infinitely near point of Γ over $E^\sim$.

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(*) A.M.S. Subject Classification (1980): 14-B-05, 32-C-40, 32-C-45 58-A-30, 58-F-14, 58-F-18.