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HARDY SPACES OF BANACH-SPACE-VALUED FUNCTIONS DEPENDING ON THE GEOMETRY OF THE BANACH SPACE

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The purpose of this note is to announce some results related to Hardy spaces of vector valued functions and to show that some properties on B have to be required if we want that the classical theorems remain valid in the B-valued setting.

Spaces of B-valued holomorphic functions.

Let B be a complex Banach space and $1 \leq p \leq \infty$. We shall denote by $H^p_B(D)$ the space of B-valued holomorphic functions on the disc D such that $\sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} \|F(r.e^{it})\|^p_B \, dt \right)$ is finite (with the obvious modification for $p = \infty$).

It is known that the necessary and sufficient condition on B so that every function F in $H^p_B(D)$ has limits at the boundary of D, $\mathbb{T}$, is the so-called analytic Radon-Nikodym property (see [3]). We shall find the boundary values space of $H^p_B(D)$ without any condition on B.

Denoting by V^p_B, $1 < p \leq \infty$, the space of B-valued measures on $\mathbb{T}$ with bounded p-variation (see [7]), and by M_B the space of B-valued regular measures with bounded variation, we have the following

THEOREM 1.- *Let* B *be a complex Banach space and* $1 < p \leq \infty$.

(1) $H^1_B(D) = \{ \mu \in M_B : \hat{\mu}(n) = 0 \text{ for } n<0 \}$

(2) $H^p_B(D) = \{ \mu \in V^p_B : \hat{\mu}(n) = 0 \text{ for } n<0 \}$

where both isometries are given by the Poisson integral.

Spaces of B-valued harmonic functions.

Let B be a real Banach space and $1 \leq p \leq \infty$. We shall denote by $ReH^p_B(D)$ the space of B-valued harmonic functions on D which are real parts of some function belonging to H^p_{B+iB}. Let us comment the connection between these spaces and the conjugate function. We shall write $\mathcal{H}^p_B = \{ f \in L^p_B : \tilde{f} \in L^p_B \}$ where $\tilde{f}$ is the conjugate function of f.

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It is well known that in the scalar-valued case this is the space of the boundary values of functions in $\mathrm{Re}H^p(D)$, but in our context we have

THEOREM 2.- *Let* B *be a real Banach space and* $1 < p \leq \infty$. *The following statements are equivalent*

(1) $B + iB$ *has the analytic Radon-Nikodym property*

(2) $\mathcal{H}^p_B$ *is isometric to* $\mathrm{Re}H^p_B(D)$ *via Poisson integral.*

Since a measure μ in M_B can be interpreted as a "B-valued distribution" ϕ_μ in $\mathcal{D}'_B$ (here we denote by $\mathcal{D}'_B$ the space $L(C^\infty(\mathbb{T}),B)$) and for every $\psi \in C^\infty(\mathbb{T})$ we have $\tilde{\psi} \in C^\infty(\mathbb{T})$ then we can define the conjugate of μ as the distribution $\tilde{\mu}$ such that $\tilde{\mu}(\psi) = -\int \tilde{\psi}(t)\,d\mu(t)$ for every ψ in $C^\infty(\mathbb{T})$. With these notations we can stablish the following

THEOREM 3.- *Let* B *be a real Banach space and* $1 < p \leq \infty$

(1) $\mathrm{Re}H^1_B(D) = \{\mu \in M_B : \tilde{\mu} \in M_B\}$

(2) $\mathrm{Re}H^p_B(D) = \{\mu \in V^p_B : \tilde{\mu} \in V^p_B\}$

where both isometries are given by the Poisson integral.

As it happens in the classical case we can consider spaces defined by means of maximal functions (see [5]). Here we shall denote by $H^1_{max,B}$ the space of functions f in L^1_B such that $\sup_{0<r<1} \|f * P_r(t)\|_B = f^*(t)$ belongs to L^1, and given a B-valued harmonic function on D, F, we shall denote by F^* its radial maximal function.

THEOREM 4.- *The following statements are equivalent*

(1) B *has the Radon-Nikodym property*

(2) *Every* B*-valued harmonic function* F *with* F^* *in* L^1 *is the Poisson integral of some* f *in* $H^1_{max,B}$.

(3) *For all* p, $1 < p \leq \infty$, *every* B*-valued harmonic function* F *with* F^* *in* L^p *is the Poisson integral of some* f *in* L^p_B.

Let us recall that a Banach space is said to have the <u>UMD property</u> if there exist p, $1 < p < \infty$, and C_p such that $\|\tilde{f}\|_p \leq C_p . \|f\|_p$ for every $f \in L^p_B$ (see [2], [4]). We can extend this result as follows

THEOREM 5.-

B has the UMD property if and only if $H^1_{max,B} = \mathcal{H}^1_B$.

Hardy spaces for $0 < p < 1$.

Let us remember that when $0 < p < 1$ the space of distributions appears as the space of boundary values in the classical case. There are some different definitions of Hardy spaces for $0 < p < 1$,(all these are equivalent for real-valued functions), which can be considered in the B-valued setting.
So a B-valued distribution ϕ is said to belong to

a) $H^p_{max,B}$ if its radial maximal function $\phi^* \in L^p$ (see [5]).

b) $H^p_{at,B}$ if $\phi = \sum \lambda_k . a_k$, a_k (p,B)-atoms and $\sum |\lambda_k|^p < \infty$ (see [6]).

c) $\mathcal{H}^p_B$ if $\mathcal{P}(\phi + i\tilde{\phi}) \in H^p_{B+iB}(D)$, where
$\mathcal{P}(\phi + i\tilde{\phi})(r.e^{it}) = (\phi + i\tilde{\phi})(P_r(.-t))$.

All these spaces are endowed with the natural norms and we can state the following relationship among them

THEOREM 5.-([1]) *For $0 < p \leq 1$,*
B has the Radon-Nikodym property if and only if $H^p_{at,B} = H^p_{max,B}$.

THEOREM 6.- *For $0 < p \leq 1$,*
B has the UMD property if and only if $H^p_{at,B} = \mathcal{H}^p_B$.

Questions related to duality and interpolation are going to be considered in a "forthcoming" paper.

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