EXTRACTA MATHEMATICAE 1, 41-43 (1986)

SEMICONTINUITY FOR THE LOCAL HILBERT FUNCTION

Juan B. Sancho de Salas

Dpto. de Matemáticas. Facultad de Ciencias.
Avda. de Elvas s.n., 06071-Badajoz.

1. Let A be a noetherian local ring with maximal ideal M . The Samuel and Hilbert functions of A are defined by

$$S(n)=\text{length } A/M^{n+1}$$

$$H(n)=\text{length } M^{n}/M^{n+1}$$

for $n\geq 0$. When A is the local ring of a noetherian scheme X at a point x , these functions are denoted $S_{X,x}$ and $H_{X,x}$ respectively.

(1.1) The Samuel function is a polynomial for all sufficiently large n. This is:

$$S(n) = a_d\binom{n}{d}+ \dots +a_1\binom{n}{1}+a_0 \qquad (n\gg 0)$$

for some integer numbers $a_0,a_1,\dots,a_d\neq 0$.

The degree d of the above polynomial is the dimension of A . Moreover, the leading coefficient a_d is the multiplicity of A .

(1.2) In order to give a precise statement of the semicontinuity theorem, one finds it necessary to introduce the difference operator Δ . Precisely, for every function f(n) one defines:

$$(\Delta f)(n)=f(n)-f(n-1) \text{ if } n\geq 1, \text{ and } (\Delta f)(0)=f(0)$$

$$(\Delta^{-1}f)(n)=\sum_{i=0}^{n} f(i)$$

so that $f=\Delta(\Delta^{-1}f)=\Delta^{-1}(\Delta f)$ and the Hilbert function of a local ring is the difference of the Samuel function. Moreover, if $f\leq g$ ($f(n)\leq g(n)$ for all n) then $\Delta^{-1}f \leq \Delta^{-1}g$, but the inequality $\Delta f\leq\Delta g$ may be false.

AMS clas. 14B05

2. Bennett [1] proved a semicontinuity theorem for the Samuel functions of the points of X=Spec(A) when A is an excellent local ring. He proved that if x is the closed point of X and y is the generic point of an r-dimensional subvariety of X , then we have:

$$\Delta^{-r} s_{X,y} \leq s_{X,x}$$

The semicontinuity theorem for the Hilbert function does not follow directly from the analogous result for the Samuel function. However, given a result of Singh [3], the semicontinuity for Hilbert functions can be proved following Bennett's proof for Samuel functions. One gets $\Delta^{-r} H_{X,y} \leq H_{X,x}$

$$\Delta^{-r+1} s_{X,y} \leq \Delta s_{X,x}$$

(2.1) We give a direct proof of the following improved version of the semicontinuity theorem:

TEOREMA 1. Let A be a noetherian local ring and let x be the closed point of X=Spec(A) . If y is the generic point of an excellent r-dimensional integral closed subscheme of X , then we have:

$$s_{X,y} \leq \Delta^{r} s_{X,x}$$

(2.2) As a consequence of this theorem we obtain the Singh's result: The Hilbert function does not increase in a permissible blowing-up.

(2.3) The proof of the theorem is based on the following crucial fact:

LEMMA. Let S(n) be the Samuel function of a noetherian local ring A with maximal ideal M , let P be a prime ideal such that A/P is a regular ring of dimension 1 and let t be an element of A such that M=P+(t) . If $\bar{S}(n)$ is the Samuel function of A/(t) and $S_P(n)$ is the Samuel function of A_P , then we have

$$\bar{S}(n) = \Delta S(n) + \mathrm{length}_A\left[\frac{(P^{n+1}:t)}{(P^{n+1}:t)\quad M^{n+1}}\right]$$

$$\bar{S}(n) = S_P(n) + \mathrm{length}_A(P^{n+1}:t)/P^{n+1}$$

(2.4) Finally, we also prove that the Hilbert function stabilizes in any sequence of permissible blowing-ups. A Blowing-up $X' \longrightarrow X$ is said to be permissible if its center Y is regular and X is normally flat along Y, i.e., the graded O_X/P-algebra $Gr_P(O_X)$ is flat, where $P \subseteq O_X$ is the ideal defined by Y.

THEOREM 2. Let x be a closed point of a noetherian scheme X and let $\ldots \longrightarrow X_m \longrightarrow X_{m-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_o = X$ be an infinite sequence of permissible blowing-ups. If (x_m) is a sequence of closed points such that $x_o = x$ and x_m is over x_{m-1}, then there exists an index s such that

$$H_{X_m, x_m} = H_{X_s, x_s}$$

for all $m \quad s$.

(2.5) This result is a crucial step towards the resolution of singularities of arbitrary excellent schemes by Hilbert functions, instead of the numerical characters used by Hironaka [2]. It reduces the problem to the nonexistence of infinite sequences of permissible blowing-ups that does not modify the Hilbert function of a given point (at least when thecenters are chosen in a convenient way). This situation should be handled with a suitable theory of maximal contact for excellent schemes that might control the infinitely near points of a given singularity.

REFERENCES

[1] B.M. Bennett: On the characteristic function of a local ring, Ann. of Math. 91(1970),25-87.

[2] H. Hironaka: Resolution of singularities of an algebraic variety of characteristic zero, Ann. of Math. 79(1964).

[3] B. Singh: Effect of a permissible blowing-up on the local Hilbert functions, Invent. Math. 26(1974),201-212.