EXTRACTA MATHEMATICAE 1, 50-52 (1986)

# MATRICES WHICH COMMUTE WITH A GIVEN MATRIX UPON A SUBSPACE(*)

by

Mª Asunción Beitia(**)

## INTRODUCTION

Let $\mathbf{M_n}(\mathbb{F})$ be the vector space of the n-square matrices over $\mathbb{F}$. For nxn matrices $\mathbf{A}$ and $\mathbf{B}$ belonging to $\mathbf{M_n}(\mathbb{F})$ we define:

$$[\, A, B \,] := AB - BA \quad \text{and} \quad C(A) = \{ X \in \mathbf{M_n}(\mathbb{F}) \;/\; [\, A, X \,] = 0 \}$$

If the characteristic polynomial of $\mathbf{A}$ is linearly factorizable in $\mathbb{F}$, then the dimension of this vector subspace of $\mathbf{M_n}(\mathbb{F})$ is well known:

a) In terms of the Segre Characteristic of the matrix $\mathbf{A}$, $[(n_{1s_1}, n_{1,s_1-1}, \dots, n_{12}, n_{11}), \dots, (n_{rs_r}, n_{r,s_r-1}, \dots, n_{r2}, n_{r1})]$ :

$$\dim C(A) = \sum_{i=1}^{r} \sum_{j=1}^{s_i} (2s_i - 2j + 1) n_{ij} \qquad \text{(see [3])}$$

b) In terms of the Weyr characteristic of the matrix $\mathbf{A}$, (conjugate of Segre´s one), $[(m_{11}, m_{12}, \dots, m_{1t_1}), \dots, (m_{r1}, m_{r2}, \dots, m_{rt_r})]$:

$$\dim C(A) = \sum_{k=1}^{r} \sum_{i=1}^{t_k} m_{ki}^2 \qquad \text{(see [1])}$$

c) In terms of the degrees of invariant polynomials of $\mathbf{A}$, $n_1, \dots, n_t$:

$$\dim C(A) = \sum_{i=1}^{t} (2i-1) n_i \qquad \text{(see [2], Chapter 8)}$$

d) In terms of the conjugate partition $(k_1, \dots, k_s)$ of $(n_1, \dots, n_t)$:

$$\dim C(A) = \sum_{i=1}^{s} k_i^2$$

We study in this paper the problem of the n-square matrices $\mathbf{X}$ which commute with the given matrix $\mathbf{A}$ upon a vector subspace $\mathbf{E}$ of $\mathbb{F}^{nx1}$.

---

(*) This work has been supported in part by the Acción Integrada Hispano-Portuguesa nº 17/5, 1984

(**) Author´s address: Departamento de Matemáticas, Escuela Universitaria de Magisterio, 01006 Vitoria-Gasteiz, Alava.

AMS Classification: 15A27, 15A24

Let $\mathbf{M_E}(\mathbf{A}) = \{ X \in \mathbf{M_n}(\mathbb{F}) \ / \ [\mathbf{A}, \mathbf{X}]\mathbf{v} = 0$ for every $\mathbf{v} \in \mathbf{E} \}$

We look for lower and upper bounds for teh dimension of $\mathbf{M_E}(\mathbf{A})$ in terms of **A** and **E**.

**Problem 1.** Find lower and upper bounds for the dimension of $\mathbf{M_E}(\mathbf{A})$, when **E** is any subspace of $\mathbb{F}^{\mathbf{nx1}}$.

**Problem 2.** Find lower and upper bounds fos the dimension of $\mathbf{M_E}(\mathbf{A})$, when dim **E** = m is fixed.

## RESULTS

**Theorem 1.** Let $\mathbf{A} \in \mathbf{M_n}(\mathbb{F})$ be diagonalizable, **E** a subspace of $\mathbb{F}^{\mathbf{nx1}}$ such that dim **E** = 1. Then:

$$\dim \mathbf{M_E}(\mathbf{A}) \leq n^2-(n-k_1)$$

where $k_1$ is the number of invariant polynomials of **A** distinct from 1.

**Theorem 2.** Let $\mathbf{A} \in \mathbf{M_n}(\mathbb{F})$ be diagonalizable, and $k_1 \geq k_2 \geq \dots \geq k_s$ the multiplicities of its eigenvalues. Let **E** be a subspace of $\mathbb{F}^{\mathbf{nx1}}$ such that dim **E** = m. Let r and t the numbers verifying:

$$m > \sum_{i=1}^{r-1} k_i \ , \qquad m \leq \sum_{i=1}^{r} k_i$$

and $n - k_i < m$ $(i=1,\dots,t)$ , $n - k_i \geq m$ $(i=t+1,\dots,s)$

Then:

$$n^2-nm + \sum_{i=1}^{t} k_i^2 -(n-m) \sum_{i=1}^{t} k_i \leq \dim\mathbf{M_E}(\mathbf{A}) \leq n^2-m(n-k_r) - k_r \sum_{i=1}^{r-1} k_i + \sum_{i=1}^{r-1} k^2$$

To prove these theorems we use the following:

**Lemma 1.** Let **E** be the space spanned by the column vectors $\mathbf{v_1},\mathbf{v_2},\dots,\mathbf{v_m}$. Let **V** be the matrix having $\mathbf{v_1},\mathbf{v_2},\dots,\mathbf{v_m}$ as columns. Then:

$$(\mathbf{AX} - \mathbf{XA})\mathbf{V} = 0 \iff (\mathbf{I_n} \otimes \mathbf{V}^t)(\mathbf{A} \otimes \mathbf{I_n} - \mathbf{I_n} \otimes \mathbf{A}^t)\mathbf{x} = 0$$

where **x** denotes $\mathrm{vec}(\mathbf{X}) = (x_{11},x_{12},\dots,x_{1n},\dots,x_{n1},x_{n2},\dots x_{nn})^t$ and $\otimes$ denotes the Kronecker product. **A** and **X** are n-square matrices belonging to $\mathbf{M_n}(\mathbb{F})$.

**Lemma 2.** Let $\mathbf{A} \in \mathbb{F}^{\mathbf{mxn}}$ be a matrix partitioned as follows: $\mathbf{A} = (\mathbf{A_1},\mathbf{A_2},\dots,\mathbf{A_s})$ where $\mathbf{A_i} \in \mathbb{F}^{\mathbf{mxk_i}}$ $i=1,\dots,s$ $\sum_{i=1}^{n} k_i = n$, $m \leq n$. We suppose that $r(\mathbf{A}) = m$ (where r denotes the rank ). If we set $\mathbf{C}_j=(\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{O}_j,\ldots,\mathbf{A}_s)$ where $\mathbf{O}_j \in \mathbb{F}^{m\times k_j}$, $j=1,\ldots,s$, then:

$$\sum_{i=1}^{s} r(\mathbf{C}_i) \geq (s-1)m$$

**Remarks.** Theorem 1 solves Problem 1 for the case of diagonalizable matrices, since if $\mathbf{E}_1$ is a subspace of $\mathbf{E}_2$, then $\mathbf{M}_{\mathbf{E}_2}(\mathbf{A}) \subseteq \mathbf{M}_{\mathbf{E}_1}(\mathbf{A})$, and so, $\dim \mathbf{M}_{\mathbf{E}_2}(\mathbf{A}) \leq \dim \mathbf{M}_{\mathbf{E}_1}(\mathbf{A})$. Thus, if $\mathbf{A}$ is a diagonalizable matrix, for every subspace $\mathbf{E}$ of $\mathbb{F}^{n\times 1}$, the following relations holds:

$$\dim \mathbf{C}(\mathbf{A}) \leq \dim \mathbf{M}_{\mathbf{E}}(\mathbf{A}) \leq n^2-(n-k_1)$$

Theorem 2 solves Problem 2 for the case of diagonalizable matrices, and if m=n, we have t=s and r=s, so:

$$\dim \mathbf{M}_{\mathbb{F}^{n\times 1}}(\mathbf{A}) \geq \sum_{i=1}^{s} k_i^2 \text{ and } \dim \mathbf{M}_{\mathbb{F}^{n\times 1}}(\mathbf{A}) \leq nk_s + \sum_{i=1}^{s-1} k_i^2 - k_s \sum_{i=1}^{s-1} k_i = \sum_{i=1}^{s} k_i^2$$

Then, $\dim \mathbf{M}_{\mathbb{F}^{n\times 1}}(\mathbf{A}) = \sum_{i=1}^{s} k_i^2$. This formula coincides with the expression given in d), since if the matrix is diagonalizable, $k_1,\ldots,k_s$ the multiplicities of its eigenvalues, and $n_1,\ldots,n_t$ the degrees of its invariant polynomials, then $(k_1,\ldots,k_s)$ is the conjugate partition of $(n_1,\ldots,n_t)$.

**References**


[1] W. Brandenbusch, Die Anzahl linear unabhängiger Matrizen X, die mit einer bestimnten Matrix A kommutieren, ausgedrückt in den Weyrschen Charakteristiken, *ZAMM* 60, 205 (1980)

[2] F.R= Gantmacher, *Théorie des Matrices* tome 1, Dunod, París (1966)

[3] J. M. Gracia, Dimension of the solution Spaces of the Matrix Equations $[A,[A,X]]=0$ and $[A,[A,[A,X]]]=0$, *L.M.A.* 9 (1980), 195-200

[4] G. Marsaglia and G.P.H. Styan, Equalities and Inequalities for Ranks of Matrices, *L.M.A.* 2 (1974), 269-292


THIS PAPER IS TO APPEAR IN PORTUGALIAE MATHEMATICA.

Vitora-Gasteiz, December 17, 1985